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On the Effects of Noise and Drift on Diffusion in Fluids

A. Crisanti[1] **and A. Vulpiani**[1]

We discuss some aspects of the intriguing problem of interplay between molecular diffusion and the geometry of the velocity field in the diffusion of test particles. By simple arguments one can understand how the diffusion coefficient can have a large enhancement from the combined effects of the noise and the drift terms in the Langevin equation ruling the motion of test particles. The same effects give rise to the superdiffusive transport observed in media with correlated random velocity fields.

KEY WORDS: Langevin equation; diffusion; superdiffusive transport.

1. INTRODUCTION

The understanding of flow-assisted diffusion is of theoretical and practical importance in various fields of science and engineering, ranging from mass and heat transport in geophysical flows to chemical engineering.[1] The dispersion of a contaminant in a fluid is the result of two different effects: advection and molecular diffusion, and, in general, it is much faster than expected by considering only the latter. At the fundamental level it is of interest to understand the mechanisms that lead to transport enhancement as a fluid is driven farther from the motionless state. The main reason for the enhancement is that transport is affected by the trajectories of individual fluid elements, or tracers, which can be quite complex even in simple laminar flows.[2]

Taking into account molecular diffusion, one can describe the motion of a fluid element by a Langevin equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t) + (2\chi)^{1/2}\, \boldsymbol{\eta}(t) \tag{1.1}$$

[1] Dipartimento di Fisica, Università di Roma "La Sapienza," I-00185 Rome, Italy.

where $\mathbf{u}(\mathbf{x}, t)$ is the Eulerian flow velocity field at the position $\mathbf{x}$ and time t, and $\boldsymbol{\eta}$ is a Gaussian white noise with zero mean and

$$\langle \eta_i(t)\, \eta_j(t') \rangle = \delta_{ij} \delta(t - t') \tag{1.2}$$

The coefficient χ is the (bare) molecular diffusion coefficient. It is known that transport and diffusion properties are affected by the presence of Lagrangian chaos, i.e., the chaotic motion of a fluid element moving according to the equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}, t) \tag{1.3}$$

In most cases the physics of passive diffusive transport can be characterized in terms of an effective diffusion coefficient which contains the cumulative effects of advection and molecular diffusion. Let $\theta(x, t)$ be the concentration of tracers evolving in time according to the Langevin equation (1.1); then

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = \chi \nabla^2 \theta \tag{1.4}$$

Equation (1.4) is the Fokker–Planck equation related to (1.1).[3] In deriving (1.4) we have used the incompressibility condition $\nabla \cdot \mathbf{u} = 0$. The diffusion process takes place on time scales much longer than the characteristic microscopic time. On these time scales the evolution of $\theta(x, t)$ is dominated by weak long-wave disturbances. The equation for these slow modes can be derived by usual multiple scale or "hydrodynamic" analysis.[4] It has the form

$$\partial_t \bar{\theta} = D_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j} \bar{\theta} + O((\nabla^2 \bar{\theta})^2), \qquad i, j = 1,..., d \tag{1.5}$$

where $\bar{\theta}$ is the concentration averaged locally over a volume of linear dimension much larger than the typical length l of the velocity field. Equation (1.5) is a weak gradient expansion valid when $|\nabla \bar{\theta}|/\bar{\theta} \ll l^{-1}$. If we neglect the high-order terms, (1.5) is the diffusion equation with an effective diffusion tensor D_{ij}.

From (1.5) it follows that D_{ij} measures the spreading on very long times of a spot of tracers evolving according to (1.1). Therefore a way of computing D_{ij} is to form directly the covariance tensor of the Lagrangian motion

$$D_{ij} = \lim_{t \to \infty} \frac{1}{2t} \langle [x_i(t) - \langle x_i \rangle][x_j(t) - \langle x_j \rangle] \rangle, \qquad i, j = 1,..., d \tag{1.6}$$

Here $\mathbf{x}(t)$ is the position of a tracer at time t, and the average is taken over the initial positions or, equivalently, over an ensemble of test particles.

Equations (1.5) and (1.6) imply that the diffusion process is Gaussian, at least on large time and space scales. This is the typical situation. There exist, nevertheless, cases where anomalous diffusion is observed, i.e., $\langle [x_i(t) - \langle x_i(t) \rangle]^2 \rangle \sim t^\beta$ with $\beta > 1$ (superdiffusion) or $\beta < 1$ (subdiffusion).[5–7] The origin of such a phenomenon is in the strong correlation between $\mathbf{u}(\mathbf{x}(t))$ and $\mathbf{u}(\mathbf{x}(t+\tau))$ for large τ.

Real fluids always have a certain degree of Lagrangian chaos, e.g., in two-dimensional flows one just needs that the stream function is time dependent. The understanding of the diffusion process is therefore a hard task since it may depend in a complicated way on the detailed structure of the Eulerian velocity field. In the presence of Lagrangian chaos the diffusion properties are rather peculiar, even in the absence of molecular diffusion.[8] We do not consider here the general problem. Nevertheless, two nontrivial limit cases can be immediately distinguished:

(a) Fully developed turbulence, where the molecular effects can be ignored on a large range of scales.

(b) Fluid velocity fields where (1.3) is integrable or quasi-integrable and the degree of Lagrangian chaos is very small.

Due to the interplay of advection and molecular diffusion, the latter also can be highly nontrivial. To illustrate this point, we shall consider simple problems dealing with the diffusion of an impurity in a specified flow of a continuous medium, without considering the origin of this field.

A frequently used system for a comprehensive investigation of transport is Rayleigh–Bénard convection, because convective flows can be created ranging from time-independent spatially periodic flows on the one hand, to turbulent flows on the other. As a result, the transport rates vary over a wide range. On one side, when the fluid is motionless, the transport is due entirely to molecular diffusion ($D_{ij} \sim \chi$). In the other extreme case, i.e., turbulent flows, transport is due to advection by the flow and is often described phenomenologically as enhanced diffusion.[9]

Between these two extrema there are two important laminar regimes: a time-dependent and a time-independent regime. In the time-dependent regime, the transport is dominated by the advection of tracer particles across roll boundaries and the particle trajectories may be chaotic; thus $D_{ij} \neq 0$ even if $\chi = 0$.[10] In the time-independent regime, large-scale transport is generally due to molecular diffusion between adjacent convection rolls, so that $D_{ij} = 0$ if $\chi = 0$. However, the structure of the Eulerian velocity field can strongly modify this result, as discussed below.

This paper is organized as follows. In Section 2 we discuss the diffusive properties of (1.1). We show that they are very sensitive to the combined effects of the structure of the velocity field and molecular diffusion. In Section 3 we consider the superdiffusive motion of test particles in a special class of random velocity fields for a 2D (or 3D) layered medium with y (and z)-dependent random velocity in the x direction. Section 4 is devoted to a pedagogical study of diffusion in a 1D velocity field periodic in space and time. In this case, for a suitable range of parameters, the diffusion coefficient does not depend on the molecular diffusion coefficient, a phenomenon similar to stochastic resonance.[11] Section 5 contains a summary and conclusions.

2. STANDARD DIFFUSION IN STEADY VELOCITY FIELDS

Real fluids always have a (small) degree of molecular diffusion. For this reason it is interesting to study the extreme case of a small molecular diffusion on simple integrable Eulerian velocity fields. The relevant dimensionless parameter which measures the relative importance of advection over molecular diffusion is the Peclet number

$$\mathrm{Pe} = \frac{VL}{\chi} \tag{2.1}$$

where V is the typical velocity of the flow, L the typical length of the convective flow, e.g., the size of the vortex cell, and χ the molecular diffusion coefficient of the medium. The Peclet number can also be understood as the ratio of the convection time L/V over the diffusion time L^2/χ across a distance L.

The physically interesting case is when the local convective transport exceeds substantially the diffusion transport, i.e., for large Peclet numbers. In this context we shall analyze the widely studied two-dimensional convective velocity field given by

$$\mathbf{u} = \left(\frac{\partial}{\partial y}\psi, -\frac{\partial}{\partial x}\psi\right) = (B\cos y, A\sin x) \tag{2.2}$$

where ψ is the stream function, $\psi = A\cos x + B\sin y$.

The qualitative behavior for $|A| = |B|$ models Rayleigh–Bénard convection, as the phase space consists of square cells separated by lines (separatrices) where the rotation time diverges; see Fig. 1a. It is clear that dispersion of a passive impurity on a large scale is impossible without the molecular diffusion that allows the jumping among different rolls. For

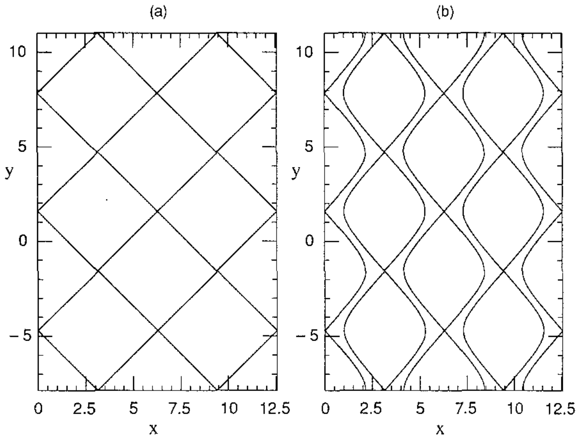

Fig. 1. Structure of the separatrices of Eq. (1.3), for the field (2.2); (a) $A = -1$, $B = 1$; (b) $A = -1.3$, $B = 1$.

a large Peclet number the effective diffusion coefficient has been found —theoretically[12] and experimentally[13]—to scale as

$$D_{ij} \sim \chi \, \mathrm{Pe}^{1/2} \sim \chi^{1/2} \tag{2.3}$$

D_{ij} is thus much larger than χ for $\chi \to 0$. A simple way of understanding (2.3) is the following. In the vicinity of the separatrix between two rolls the component of the flow perpendicular to the separatrix vanishes, and the only mechanism of transport from one roll to another is by molecular diffusion. Therefore the only particles which can leave the roll, and contribute to transport, are those "not too far" from the boundary. All the others have not enough time to diffuse through the separatrix. As a consequence, the transport is entirely due to particles in a layer of width δ near the separatrix. Therefore one can estimate D by simply noting that the particles close to the separatrices perform a random walk with diffusion coefficient $\sim L^2/\tau$, where $\tau = L/V$. The fraction of particles in the "active" layer is $\sim \delta/L$, so that the effective diffusion coefficient is $D \sim (\delta/L)(L^2/\tau)$. The width δ depends on the molecular diffusion, and is roughly given by $\delta^2 \sim \chi\tau$. This immediately leads to the conclusion that the effective diffusion coefficient scales as $D \sim \chi \, \mathrm{Pe}^{1/2}$. This result can be obtained in a more rigorous way; the interested reader is referred to the references.

When $|A| \neq |B|$, narrow channels arise among the convective cells in a direction which depends on the relative magnitude of $|A|$ and $|B|$; see Fig. 1b. The motion of test particles inside a channel appears to be ballistic and this enormously enhances the transport along the channel direction.

The process is strongly anisotropic and can be regarded as due to long runs in the channel interrupted by trapping periods inside the rolls. We thus introduce two effective diffusion coefficients, $D_{\parallel}$ along the channels direction and $D_{\perp}$ along the direction transverse to the channels. In the limit of small molecular diffusion one has[14, 15]

$$D_{\parallel} \propto \frac{L^2}{V} \,||A| - |B||^3 \chi^{-1} \qquad \text{and} \qquad D_{\perp} \propto V \,||A| - |B||^{-1} \chi \tag{2.4}$$

As in the case $|A| = |B|$, (2.4) can be derived by simple arguments. Without losing in generality, by a suitable choice of length and time units, we have $L = O(1)$ and $V = O(1)$, so that we can set $B = 1$ and $A = -(1 + \delta)$. In the following our dimensional arguments neglect multiplicative factors $O(1)$, such as L and V. The stream function becomes $\psi = \sin y - (1 + \delta) \cos x$, which for $\delta = 0$ describes convection cells of width 2π, where in the absence of a noise term the motion of a test particle is always periodic. The separatrices are the lines where the stream function is zero (for $\delta = 0$) and cross at the unstable hyperbolic fixed points of the flow. When $\delta > 0$ the borderlines between cells do not coincide and there appear channels along the y direction; see Fig. 1b. By simple perturbative calculations one finds that for small δ the width of a channel is $\sim \delta$, although the maximum distance between the separatrices increases up to $\sim \delta^{1/2}$ near the unstable fixed points. The case $\delta < 0$ corresponds to channels along the x axis. The motion of a particle inside the channels, neglecting molecular diffusion, is ballistic and the velocity field changes sign between neighboring channels. For small χ, a test particle can jump into a channel, because of molecular diffusion. Then, one has a ballistic motion inside the channel with velocity $V_c \sim O(1)$ either in the up or in the down y direction stopped by a capture from a cell after a time $T_c \sim \delta^2/\chi$, and so on. Let us consider the case for which

$$T_c/T_r \gg 1, \qquad \text{i.e.,} \quad \delta^2/\chi \gg 1$$

since the circulation time $T_r \sim V/L \sim O(1)$. With this dimensional estimate of the diffusive times in the channels we can compute the effective diffusivity tensor, which in these coordinates is diagonal with $D_{\perp} = D_{11}$ and $D_{\parallel} = D_{22}$. The typical length of a run along a channel is

$$L_c \sim T_c V_c \sim \delta^2/\chi \tag{2.5}$$

The probability p to find a particle in a channel is proportional to its width $\sim \delta$, and thus

$$D_{\parallel} \sim p \frac{L_c^2}{T_c} \sim \frac{\delta^3}{\chi} \tag{2.6}$$

On the other hand, the transport in the x direction can be described as a random walk where the time step is T_c and the length step is the cell width $\sim 2\pi$. This leads to

$$\frac{\langle [x(t) - \langle x \rangle]^2 \rangle}{4\pi^2} \propto p \frac{t}{T_c} \tag{2.7}$$

so that

$$D_\perp \sim \frac{\chi}{\delta} \tag{2.8}$$

These arguments are valid only in the limit of large T_c/T_r. By this we mean that the time spent in the channels should be large with respect to the circulation time T_r, i.e., $\chi \ll \delta^2$. When T_c/T_r becomes smaller than unity, a particle does not have enough time to perform a significant run along a channel between two successive trappings. Practically the transport process can be described as if there were no channels. In this limit, $\chi \to 0$, $\delta \to 0$, with $T_c/T_r \sim 1$, the anisotropy disappears and one recovers the $|A| = |B|$ scaling (2.3),

$$D_{||} \sim D_\perp \propto (VL)^{1/2} \chi^{1/2} \tag{2.9}$$

The agreement with the numerical data is very good for $D_{||}$, but only fair for $D_\perp$ (see Figs. 2 and 3). This is because the above scaling arguments

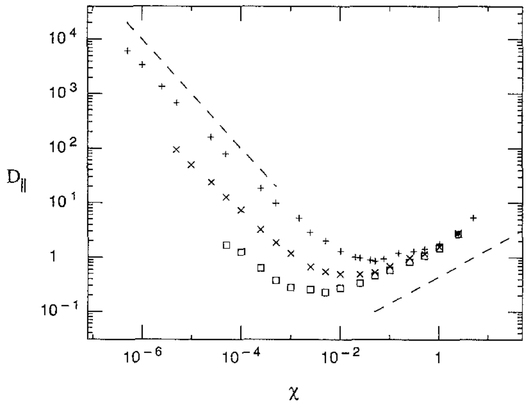

Fig. 2. The longitudinal diffusion coefficient versus the molecular diffusion coefficient for the field (2.2) and $|A| - |B| = 0.30$ (+), 0.15 (×), 0.075 (□). The dashed lines with slopes -1 and 1/2 are drawn for comparison. The numerical error bars are comparable with the symbol size.

do not consider the additional linear term in χ due to the bare molecular diffusion. We stress that in spite of the apparently "anomalous" diffusion process—long runs interrupted by trappings (see Fig. 4)—for large t the kurtosis of $[x(t) - \langle x \rangle]$ tends to the Gaussian value 3 so that the diffusion is standard and Gaussian.

The situation here is similar to the one obtained for the truncated Navier–Stokes equations.[8] There the "jumping" was due to the Lagrangian chaos, here to the molecular diffusion. However, the physical mechanisms which rule the diffusion are the same. The tracer is trapped for long times in a small limited region of space and then escapes along ballistic channels until a subsequent trap.

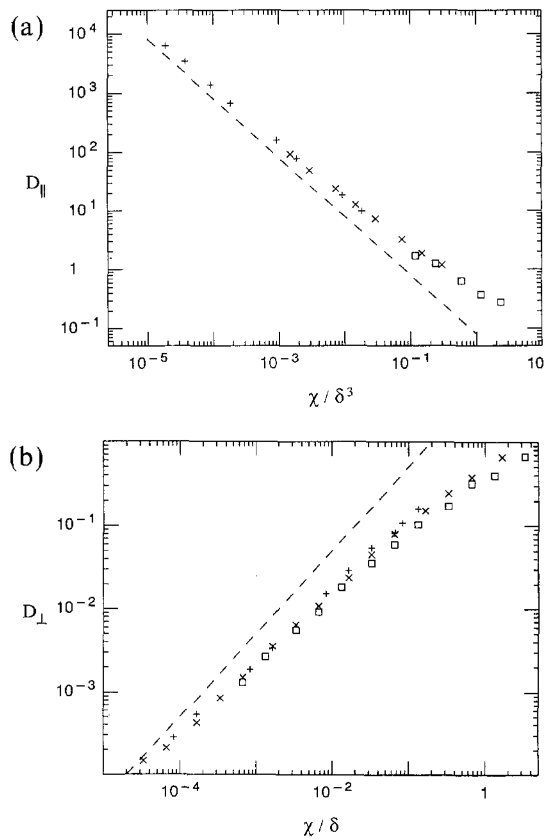

Fig. 3. (a) The longitudinal diffusion coefficient versus χ/δ^3 (together with a dashed line of slope -1). (b) The transverse diffusion coefficient versus χ/δ (together with a dashed line of slope -1). The symbols refer to $\delta = |A| - |B| = 0.30$ (+), 0.15 (×), 0.075 (□). The numerical error bars are comparable with the symbol size.

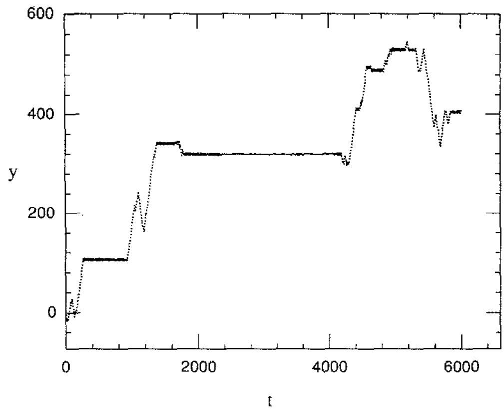

Fig. 4. Plot of y versus t for one particle driven by Eq. (1.1) with the velocity field of Fig. 1(b) and $\chi = 0.001$.

The above analysis was performed for two-dimensional systems; however, it is easy to realize that similar behaviors may be also found in three-dimensional ones. This is due to the ballistic channels which appear in a large class of systems. For example, model (2.2) presents the same behavior as the dispersion of a contaminant for laminar flows in long straight tubes or channels. By the direct analysis of Eq. (1.4) for a passively advected scalar field, Taylor[16] has shown that there is a strong enhancement of the longitudinal diffusion coefficient $D_{\parallel}$, while the radial inhomogeneities are smoothed out:

$$D_{\parallel} = \frac{\hat{\delta}}{\chi} \frac{U^2}{48} + \chi \tag{2.10}$$

where $\hat{\delta}$ is the radius of the tube and U the average velocity of the flow. The first term of the r.h.s. of (2.10) always dominates, because $\hat{\delta} U^2/\chi^2$ is very large ($\gg 1$), except for the case of very slow flows and/or extremely fine capillaries. The presence of transverse velocity gradients causes a sharp increase of $D_{\parallel}$. It is worth stressing that the larger the molecular diffusion, the smaller the longitudinal dispersion. The dependence on $\hat{\delta}^2$ instead of δ^3 is due to the fact that in (2.10) all the particles contribute to the diffusion.

We have discussed only one type, even if quite general, of convective velocity field. The analysis leads to power-law dependence of the effective diffusion coefficient on the Peclet number. This is a quite general result. Indeed in the limit $\mathrm{Pe} \gg 1$ it is natural to assume a power-law scaling

$$D \sim \chi \, \mathrm{Pe}^{\alpha}, \qquad \mathrm{Pe} \gg 1 \tag{2.11}$$

with an exponent α that depends on the topology of the velocity field. It is clear that the most unfavorable situation is for a plane divided into finite convective cells, as in the above flow for $|A| = |B|$. In this case we have $\alpha = 1/2$. The other extreme case is when there is a finite fraction of streamlines which go to infinity in some directions, e.g., the channels in the above example. In this case $\alpha = 2$. Therefore, from a physical point of view, we expect $1/2 \leqslant \alpha \leqslant 2$. For example, in the so-called "common-position" case one finds $\alpha = 10/13$.[17]

3. ANOMALOUS DIFFUSION IN RANDOM VELOCITY FIELDS

Consider a 2D velocity field of the form

$$\mathbf{u} = (u(y), 0) \tag{3.1}$$

where $u(y)$ is a quenched random function with a given spectrum $S(k)$, i.e.,

$$u(y) = \int_{-\infty}^{+\infty} dk\, e^{iky} U(k), \qquad \overline{U(k)\, U(k')} = S(k)\, \delta(k - k') \tag{3.2}$$

The average is over the realizations of the field.

Matheron and De Marsily[7] showed that anomalous diffusion in the x direction occurs if

$$\int_0^{\infty} dk\, \frac{S(k)}{k^2} = \infty \tag{3.3}$$

For example, in the case $S(k) = \text{const}$, i.e., the velocity field is spatially a white noise, one obtains

$$\langle \overline{[x(t) - x(0)]^2} \rangle \sim t^{3/2}$$

In this section we rederive these results in a very simple way, and discuss the generalization to the 3D case and its connection with the problem treated in Section 2.

With a velocity field $\mathbf{u}$ of the form (3.1) one has from (1.1)

$$\frac{dx}{dt} = u(y(t)) + (2\chi)^{1/2}\, \eta_1$$

$$\frac{dy}{dt} = (2\chi)^{1/2}\, \eta_2$$

which can be formally integrated, yielding

$$y(t) - y(0) = (2\chi)^{1/2} \int_0^t dt' \, \eta_2(t') \tag{3.4}$$

and

$$\begin{aligned} x(t) - x(0) &= \int_0^t dt' \, u(y(t')) + (2\chi)^{1/2} \int_0^t dt' \, \eta_1(t') \\ &= \int_0^t dt' \int_{-\infty}^{+\infty} dk \, e^{iky(t')} U(k) + (2\chi)^{1/2} \int_0^t dt' \, \eta_1(t') \end{aligned} \tag{3.5}$$

Taking the square of (3.5) and averaging over η and over the realizations of the velocity field, one has

$$\langle \overline{[x(t) - x(0)]^2} \rangle = \int_0^t dt' \int_0^t dt'' \int_0^\infty dk \, e^{-(k^2/2)\chi|t' - t''|} S(k) + 2\chi t \tag{3.6}$$

In deriving (3.6), we made use of the well-known result for a Gaussian process w with zero mean,

$$\overline{\exp(iw)} = \exp(-\tfrac{1}{2}\overline{w^2})$$

Introducing the variables $t_1 = t' - t''$ and $t_2 = t''$ and neglecting the term $2\chi t$ in (3.6) leads to

$$\langle \overline{[x(t) - x(0)]^2} \rangle \sim \int_0^t dt_2 \int_0^{t_2} dt_1 \int_0^\infty dk \exp\left(-\frac{k^2}{2}\chi t_1\right) S(k) \tag{3.7}$$

The leading contribution comes from the $k \to 0$ part. In the case of $\int dk \, k^{-2} S(k) < \infty$ the diffusion is standard with

$$D \sim \frac{1}{\chi} \int_0^\infty dk \, \frac{S(k)}{k^2} \tag{3.8}$$

If for simplicity we assume that

$$S(k) \sim k^\gamma \qquad \text{for} \quad k \to 0 \tag{3.9}$$

then standard diffusion occurs if $\gamma > 1$. On the other hand, if $-1 \leqslant \gamma \leqslant 1$, the particles will perform anomalous diffusion with

$$\langle \overline{[x(t) - x(0)]^2} \rangle \sim t^{2\nu}, \qquad \nu = \frac{3 - \gamma}{4} \tag{3.10}$$

This result is easily obtained by inserting (3.9) into (3.7) and using the auxiliary variable $\tau = k^2 \chi t$.

Note that $\int dk\, k^{-2} S(k) \sim \overline{u^2} \delta^2$, where δ is the typical length of the velocity field $u(y)$, i.e., the typical distance between two zeros of $u(y)$. Therefore (3.8) is essentially the result (2.6) or (2.10). For $\gamma = 0$, (3.10) gives $\nu = 3/4$, a result previously obtained by many authors.[18]

If (3.9) holds only for $k \geqslant k_0$ while $S(k) = 0$ for $k < k_0$ it is possible to show[19] that the anomalous diffusion law (3.10) is valid only for $t \leqslant 1/(k_0^2 \chi)$. For larger time the diffusion is standard with a diffusion coefficient given by (3.8).

The condition $\int dk\, k^{-2} S(k) = \infty$ for an anomalous diffusion can be understood with a physical argument. In fact, $\int dk\, k^{-2} S(k) < \infty$ means that the typical distance δ between to zeros of $u(y)$ is finite. In this case the process is similar to that of a velocity field given by a sequence of strips of size δ and velocity $\pm V_0$ alternatively, where V_0 is the typical velocity $[\int dk\, S(k)]^{1/2}$. One can then repeat the arguments of Section 2 and obtain (3.8).

Obviously if $\int dk\, k^{-2} S(k) = \infty$, i.e., $\delta = \infty$, the approach of Section 2 cannot be used and the problem must be treated in a more careful way.

The generalization to higher dimension is straightforward. Consider, in fact, a 3D velocity field $\mathbf{u} = (u(y, z), 0, 0)$ where

$$u(y, z) = \int d\mathbf{k}\; U(\mathbf{k}) e^{i(k_y y + k_z z)}$$

with $\mathbf{k} = (k_y, k_z)$ and $d\mathbf{k} = dk_y\, dk_z$. By introducing the spectral function $S(k)$ as

$$\overline{U(\mathbf{k})\, U(\mathbf{k}')} = \frac{S(k)}{k} \delta(\mathbf{k} - \mathbf{k}')$$

the above considerations can be straightforwardly repeated.

4. DIFFUSION IN TIME-DEPENDENT FIELDS: A PEDAGOGICAL EXAMPLE

In this section we shall discuss a pedagogical example which shows the relevance of the combined effect of noise and drift.

Consider the 1D motion of a particle subject to a drift term $-\partial V/\partial x$ and to a noise term:

$$\frac{dx}{dt} = -\frac{\partial V}{\partial x} + (2\chi)^{1/2}\, \eta \qquad (4.1)$$

The potential V is periodic in space and time

$$V(x, t) = V(x + 2\pi, t), \qquad V(x, t) = V(x, t + T)$$

To be explicit, we take the simple form

$$V(x, t) = (1 + A \sin \omega t) \cos x, \qquad \omega = 2\pi/T \tag{4.2}$$

Numerical data show that the particle performs a standard diffusion process. In general, however, the diffusion coefficient D has a nontrivial dependence on the parameters A, T, and χ.

For $A = 0$ the potential does not change in time, and hence D can be estimated for $\chi \to 0$ from the Kramer formula.[20] The particle in a minimum of V will jump in one of the two near neighbors minima in a typical time

$$\tau \sim \exp(2/\chi)$$

and thus $D \sim \tau^{-1} \sim \exp(-2/\chi)$.

If A is not very small, in general, it is not easy to have an estimate of D. However, there is a range of values of parameters where D is a function of T alone. This case is similar to stochastic resonance.[11]

In the following we shall assume $|A| > 1$. In the dynamics there are three relevant time scales.

1. The relaxation times obtained from the linearized equation of motion about the minima. For example,

$$\tau_1 = \frac{1}{|V''(x_1, t = 0)|}, \qquad \tau_2 = \frac{1}{|V''(x_2, t = T/2)|}$$

where $V''(x, t) = \partial^2 V/\partial x^2$ and x_1 and x_2 are the positions of a minimum at $t = 0$ and $t = T/2$, respectively.

2. The characteristic potential oscillation time $T/2$.

3. The characteristic jump times between two minima. Assuming a frozen potential, one has for the potential at $t = 0$ and $t = T/2$

$$T_1 = \pi(\tau_1 \tau_2)^{1/2} \exp(\Delta V_1/\chi), \qquad T_2 = \pi(\tau_1 \tau_2)^{1/2} \exp(\Delta V_2/\chi)$$

where $\Delta V_1 = 2(|A| + 1)$, $\Delta V_2 = 2(|A| - 1)$.

If two or more of these scales are of the same order of magnitude, the mechanism leading to diffusion is nontrivial. On the other hand, if

$$\tau_{1,2} \ll T/2 \ll T_{1,2} \tag{4.3}$$

it is easy to show that

$$D = \frac{\pi^2}{T} \tag{4.4}$$

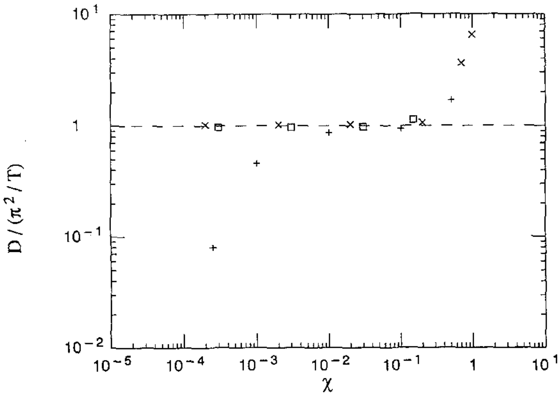

Fig. 5. Plot of $D/(\pi^2/T)$ versus χ for different values of A and T. (+) $A = 1.5$, $T = 75$; (×) $A = 1.7$, $T = 120$; (□) $A = 1.5$, $T = 300$.

In fact, if (4.3) is satisfied, the motion of the particle is essentially the following: $x(t)$ relaxes about a minimum of V in a time $\sim\tau_{1,2}$. With probability almost one, the particle remains near the minimum for a time of $O(T/2)$ since $T_{1,2} \gg T/2$. After a time interval of the order of half of the period of oscillation of V the minimum will change into a maximum. The particle will then relax very quickly, in a time $O(\tau_{1,2})$, in one of the two nearest minima placed at distance $\pm\pi$. The motion is then a random walk with a step π at each half-period. Therefore for $t \gg T$ we have

$$\overline{[x(t) - x(0)]^2} \simeq \pi^2 \left[\frac{t}{T/2}\right] = 2\frac{\pi^2}{T}t \tag{4.5}$$

from which (4.4) follows. Figure 5 shows $D/(\pi^2/T)$ versus χ for different values of A and T. The figure clearly shows that if (4.3) holds, the relation (4.4) is well satisfied. On the other hand, for $\tau_{1,2}/T$ not small enough, D depends on χ in a nontrivial way.

5. SUMMARY

We have discussed the sensitivity of the diffusive properties of test particles on the geometrical details of the velocity fields and molecular diffusion. Their combined effect is at the origin of superdiffusive transport occurring in some random correlated velocity fields.

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